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Asymptotic solutions of an elliptic equation system on a Riemannian manifold concentrated in the vicinity of a phase trajectory

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Abstract. The scalar Helmholtz equation has solutions concentrated in a small vicinity of a ray. Outside this vicinity these solutions decrease exponentially. Here solutions with similar features are constructed for an elliptic equation system and examples such as the linear problems of elasticity theory, crystallo-optics, magnetic hydrodynamics and gravitation theory are given.

1. Introduction

It is well known that a scalar Helmholtz equation $\Delta U + \omega^2 c^{-2} U = 0$ at $\omega \to \infty$ has solutions concentrated in a small vicinity (of $\omega^{-1/2}$ order of magnitude) of a ray, i.e. a characteristic of the eikonal equation $(\nabla \tau)^2 = c^{-2}$. Outside this vicinity these solutions decrease exponentially. In the present paper solutions with similar features are constructed for an elliptic equation system of a general type. Linear problems of elasticity theory, crystallo-optics, magnetic hydrodynamics and gravitation theory give particular examples of elliptic systems of that kind.

The solutions concentrated in the vicinity of a phase trajectory give the possibility of finding the asymptotics for the subsets of eigenvalues of the corresponding operators on compact manifolds. It is the most important use of that sort of solution. By integrating the solutions concentrated in the vicinity of a phase trajectory with respect to a parameter one can find the asymptotic solution of the initial system in the vicinity of the ray field singularities where the formulae of the ray method are inapplicable.

In a series of previous articles the solutions concentrated in the vicinity of the geodesic for the Laplace operator on a compact manifold were constructed (Babich and Buldyrev 1972, Babich and Lazutkin 1968). Similar solutions were obtained for the special case of magnetic hydrodynamics (Buldyrev 1971), isotropic elasticity theory (Kirpicnikova, 1971), and electrodynamics (Pankratova 1971).

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2. Formulation of the problem. Phase trajectories and linearisation of the characteristic system

Let V_{n+1} be a smooth $(n+1)$-dimension Riemannian manifold with metric tensor field $g_{ij}(x)$. We treat the elliptic system

$$\nabla_i[A^{is}_{jr}(x)\nabla_s U^r]+\omega^2 B_{jr}(x)U^r=0, \qquad j=0,1\ldots,n \tag{2.1}$$

with respect to vector field $U(x)$ in a domain $\Omega\subset V_{n+1}$. Here $A^{is}_{jr}(x)$ and $B_{jr}(x)$ are smooth real tensor fields in Ω of the 4th and 2nd rank respectively, ∇_i is an operator of covariant differentiation and ω is a parameter. Here and later on we imply summation from 0 to n over repeating indices. The tensors A^{is}_{jr} and B_{jr} are symmetrical and positive:

$$A^{is}_{jr}=A^{si}_{rj} \qquad B_{jr}=B_{rj} \tag{2.2}$$

$$A^{is}_{jr}\eta^j_i\eta^r_s\geqslant a\sum_{i,j=0}^{n}(\eta^i_j)^2 \qquad B_{jr}\xi^j\xi^r\geqslant b\sum_{j=0}^{n}(\xi^j)^2 \qquad a>0 \qquad b>0. \tag{2.3}$$

Here η^j_i and ξ^j are arbitrary real tensor fields of the 2nd and 1st rank, respectively†.

The object of this paper is the construction of the high-frequency asymptotics $\omega\to\infty$ for the vector field $U(x)$ in a vicinity of the fixed phase trajectory.

2.1. The Hamilton–Jacobi equation and phase trajectories

Let $H(x,p)$ be one of the positive roots of the equation

$$\det\|A^{is}_{jr}(x)p_ip_s-H^2B_{jr}(x)\|=0. \tag{2.4}$$

The equation

$$H(x,\nabla\tau)=1 \tag{2.5}$$

is called the Hamilton–Jacobi equation (or eikonal equation). It follows from formulae (2.2)–(2.4) that the $H(x,p)$ function is positive and homogeneous to first order with respect to p, and the matrix

$$C_{jr}(x,p)=A^{is}_{jr}(x)p_ip_s-H^2(x,p)B_{jr}(x) \tag{2.6}$$

is symmetrical.

The characteristics of equation (2.5) i.e. the curves $S=\{x^i(t),p^i(t)\}$ in the $2n+2$ dimensional phase space $\{x,p\}$, defined by the solutions of the characteristic Cauchy system

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}=\frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t}=-\frac{\partial H}{\partial x^i} \qquad 0\leqslant i\leqslant n \tag{2.7}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=p_i\frac{\partial H}{\partial p_i} \qquad H(x,p)=1 \tag{2.8}$$

are called the phase trajectories of the system (2.1). The projections of the phase trajectories on a V_{n+1} manifold are usually called rays in diffraction theory. According

† The equation (2.1) characterises the propagation of the harmonic waves with the frequency ω in the inhomogeneous anisotropic elastic medium, if we take A^{is}_{jr} as an elasticity tensor and $B_{jr}=\rho g_{jr}$ where ρ is the density of the medium.

to the Euler theorem about the homogeneous functions

$$p_i \frac{\partial H}{\partial p_i} = H \tag{2.9}$$

therefore, from (2.8) it follows that $\mathrm{d}\tau/\mathrm{d}t = 1$. It gives the possibility of parametrising the phase trajectory with the values of τ.

2.2. *Transversal stratification*

Let S be a phase trajectory and S' be the corresponding ray. If S is defined in local coordinates $\{x^0, \ldots, x^n\}$ by the equations

$$x^0 = \tau \qquad x^i = 0 \qquad p_0 = 1 \qquad p_i = 0 \qquad i = 1, \ldots, n \tag{2.10}$$

then the coordinates $\{x^0, x^1, \ldots, x^n\}$ are called transversal with respect to S'.

Lemma 1. (i) Let S be a fixed phase trajectory. Then there exist a set of maps on the manifold V_{n+1}, mapping the ray S', the local coordinates on each map being transversal with respect to S'.

(ii) In order that the curve S, defined by equations (2.10), be a phase trajectory it is necessary and sufficient that

$$\mathring{H}_{x^i} = 0 \qquad \mathring{H}_{p_i} = 0 \qquad \mathring{H} = 1 \qquad i = 1, \ldots, n. \tag{2.11}$$

Here and later on we use the notations

$$\mathring{F}(\tau) = F|S = F(x, p)\Big|_{\substack{x^0=\tau,\, x^i=0 \\ p_0=1,\, p_i=0}} \qquad F_{x^i} = \frac{\partial F}{\partial x^i} \qquad F_{p_i} = \frac{\partial F}{\partial p_i}$$

where $F(x, p)$ is a scalar or a tensor.

To each point of the ray S' there corresponds a vector $p(\tau) = (p_0(\tau), \ldots, p_n(\tau))$. The vectors orthogonal to $p(\tau)$ in the g_{ij} metric form an n-dimensional linear space $\phi_S(\tau)$. A set of pairs $\{\tau, \phi_S(\tau)\}$ is called a transversal stratification of the ray S'. The transversal stratification is an analogue to normal stratification of the geodesic for Laplace operators. From the local point of view $\phi_S(\tau)$ may be treated as an n-dimensional hyperplane on the V_{n+1} manifold, which intersects the ray S' at the point corresponding to the given value of the parameter τ. Therefore, the transversal with respect to the S' coordinates is interpreted in the following way: the equality $x^0 = \tau$ defines a point on S', and variables x^i $(1 \leqslant i \leqslant n)$ define a point on $\phi_S(\tau)$.

2.3. *Canonical system in the linear approximation*

We shall treat a linear system of equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau} y = R(\tau)q + L(\tau)y & y = (y^1, \ldots, y^n) \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}\tau} q = -L^{\mathrm{T}}(\tau)q - T(\tau)y & q = (q_1, \ldots, q_n) \end{cases} \tag{2.12}$$

where

$$R = \|\mathring{H}_{p_i p_s}\| \qquad L = \|\mathring{H}_{p_i x^s}\| \qquad T = \|\mathring{H}_{x^i x^s}\| \tag{2.13}$$

are matrices of order $n \times n$ $(1 \leqslant i, s \leqslant n)$, depending on τ and L^{T} is the matrix transposed to L.

It follows from lemma 1 that the system (2.12) is a linearisation of the canonical system (2.7) in the vicinity of the fixed phase trajectory S in coordinates transversal with respect to S'.

We call the curves $\{\tau, y(\tau)\}$ on the transversal stratification of the ray S', defined by solutions of the system (2.12), the rays in the linear approximation. Moreover the ray S' itself is defined by trivial solution $y(\tau)=0$. The rays in the linear approximation are important when concentrated solutions of the system (2.1) are constructed.

Let s be the length of the ray S', measured from the point $s=s_0$. It is easy to prove, using equations (2.7), (2.8) and (2.11), that the values s and τ on the ray S' are connected with the equality

$$s=s_0+\int_{\tau_0}^{\tau}\sqrt{g_{00}(\tau)}\,\mathrm{d}\tau.$$

3. Formal asymptotic expansion of the solution and the parabolic equation

The initial system (2.1) is invariant to the choice of the local coordinates on V_{n+1}. Let us assume that the local coordinates are transversal with respect to the ray S'. We introduce the 'reduced' coordinates

$$\nu^i=\omega^{1/2}x^i \qquad 1\leqslant i\leqslant n \qquad x^0=\tau. \tag{3.1}$$

Let us seek the vector $U(x)$ in the form of formal asymptotic expansion

$$U(\tau,\nu)=\mathrm{e}^{\mathrm{i}\omega\tau}\sum_{k=0}^{\infty}\omega^{-k/2}W_k(\tau,\nu) \qquad \omega\to\infty. \tag{3.2}$$

First we shall derive a recurrent system for $W_k(\tau,\nu)$ vectors. Let us represent the $A_{jr}^{is}(x)$ and $B_{jr}(x)$ coefficients and the Christoffel symbols Γ_{ij}^{l} in the system (2.1) by Taylor series in the vicinity of the ray S' and substitute the expansion (3.2) instead of U. By equating to zero the coefficients of the formal asymptotic series in the left-hand side of (2.1) we obtain the recurrent system for the vectors W_k:

$$\mathring{C}W_0=0 \tag{3.3}$$

$$\mathring{C}W_k=\sum_{j=1}^{k}\mathscr{L}_jW_{k-j} \qquad k=1,2\ldots. \tag{3.4}$$

The matrix C is defined by the equality (2.6) and $\mathscr{L}_j$ are the matrix differential operators of order $\min(j,2)$. The operators $\mathscr{L}_1$ and $\mathscr{L}_2$ are of the form

$$\mathscr{L}_1=\mathrm{i}\{\mathring{C}_p,\nabla\}-\{\mathring{C}_x,\nu\} \tag{3.5}$$

$$\mathscr{L}_2=\tfrac{1}{2}\{\mathring{C}_{pp}\nabla,\nabla\}+\mathrm{i}\{\mathring{C}_{px}\nu,\nabla\}-\tfrac{1}{2}\{\mathring{C}_{xx}\nu,\nu\}$$
$$+\tfrac{1}{2}\mathrm{i}\mathring{C}_{p_sx^s}+\mathring{B}\mathscr{L}+\mathrm{i}\mathring{C}_{p_0}\frac{\partial}{\partial\tau}+\frac{\mathrm{i}}{\sqrt{g}}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathring{B}\sqrt{g})+\mathrm{i}\mathring{M}(\tau). \tag{3.6}$$

Here the following notation is introduced:

$$\nu = (\nu^1, \ldots, \nu^n) \qquad \nabla = \left(\frac{\partial}{\partial \nu^1}, \ldots, \frac{\partial}{\partial \nu^n}\right) \qquad g = \det \|\mathring{g}_{ij}(\tau)\|$$

$$\{\mathring{C}_p, \nabla\} = \sum_{s=1}^{n} \mathring{C}_{p_s} \frac{\partial}{\partial \nu^s} \qquad \{\mathring{C}_{px}\nu, \nabla\} = \sum_{s,l=1}^{n} \mathring{C}_{p_s x^l} \nu^l \frac{\partial}{\partial \nu^s}$$

M is the antisymmetric matrix with components

$$M_{jr} = \frac{1}{z}\frac{\partial}{\partial x^i}(A_{jr}^{i0} - A_{rj}^{i0}) + \tfrac{1}{2}(A_{jr}^{i0} - A_{rj}^{i0})\Gamma_{li}^{l} + A_{lj}^{i0}\Gamma_{ir}^{l} - A_{lr}^{i0}\Gamma_{ij}^{l} \tag{3.7}$$

and $\mathscr{L}$ is the 'parabolic operator'

$$\mathscr{L} = 2\mathrm{i}\frac{\partial}{\partial \tau} + (R\nabla, \nabla) + 2\mathrm{i}(L\nu, \nabla) - (T\nu, \nu) + \mathrm{i}\,\mathrm{Tr}L \tag{3.8}$$

where the matrices R, L, T are defined by equalities (2.13).

Let $I_H(x, p) = \ker C(x, p)$ be the kernel of the operator C (2.6). We assume that S is a phase trajectory of a constant multiplicity, i.e. the multiplicity of the root $H(x, p)$ of the equation (2.4) and hence the dimension of the subspace $I_H(x, p)$ remains constant on S.

Lemma 2. Let $J_H(\tau, \nu)$ be a subspace of vectors which can be represented in the form $w(\tau, \nu)\mathring{\psi}(\tau)$ where $w(\tau, \nu)$ is an arbitrary smooth function, $\psi \in I_H$, and $J_H^{\perp}(\tau, \nu)$ is its orthogonal complement in the $\mathring{g}_{ij}$ metric. Then

$$\mathscr{L}_1 J_H \subset J_H^{\perp}.$$

Proof. Taking into account the identity

$$C(x, p)\psi(x, p) = 0 \qquad \psi \in I_H \tag{3.9}$$

we obtain

$$\mathscr{L}_1(w\mathring{\psi}) = [\mathrm{i}\{\mathring{C}_p\mathring{\psi}, \nabla\} - \{\mathring{C}_x\mathring{\psi}, \nu\}]w = -\mathring{C}[\mathrm{i}\{\mathring{\psi}_p, \nabla\} - \{\mathring{\psi}_x, \nu\}]w \in J_H^{\perp} \tag{3.10}$$

because the matrix C is symmetric.

It follows from equation (3.3) that

$$W_0 \in J_H \qquad W_0(\tau, \nu) = w_0(\tau, \nu)\mathring{\psi}(\tau) \qquad \psi \in I_H. \tag{3.11}$$

We seek the vectors W_k for $k \geqslant 1$ in the form

$$W_k(\tau, \nu) = w_k(\tau, \nu)\mathring{\psi}_k(\tau) + W_k^{\perp}(\tau, \nu) \tag{3.12}$$

where $\psi_k \in I_H$, $W_k^{\perp} \in J_H^{\perp}$. In general the inhomogeneous equations (3.4) are unsolvable with respect to W_k. It is necessary and sufficient for their solvability that

$$\sum_{j=1}^{k} \mathscr{L}_j W_{k-j} \in J_H^{\perp} \qquad k = 1, 2 \ldots. \tag{3.13}$$

It follows from the equality (3.11) and lemma 2 that the equation (3.4) for $k = 1$ is solvable, and

$$W_1^{\perp} = \mathring{C}^{-1}\mathscr{L}_1 W_0 \tag{3.14}$$

where $\mathring{C}^{-1}$ is the inverse operator of $\mathring{C}$ on the subspace $J_H^\perp$. Taking into account (3.12) and (3.14) the condition (3.13) for $k=2$ takes the form

$$\mathscr{L}_1 W_1+\mathscr{L}_2 W_0=\mathscr{L}_1(w_1\mathring{\psi}_1)+(\mathscr{L}_1\mathring{C}^{-1}\mathscr{L}_1+\mathscr{L}_2)W_0\in J_H^\perp$$

hence

$$(\mathscr{L}_1\mathring{C}^{-1}\mathscr{L}_1+\mathscr{L}_2)W_0\in J_H^\perp \tag{3.15}$$

because $\mathscr{L}_1(w_1\mathring{\psi}_1)\in J_H^\perp$ from lemma 2. The condition (3.15) makes it possible to obtain the vector W_0. It is easy to prove that (3.15) is equivalent to the condition

$$-\mathscr{L}_1[\mathrm{i}\{\mathring{\psi}_p, \nabla\}-\{\mathring{\psi}_x, \nu\}]w_0+\mathscr{L}_2(\mathring{\psi}w_0)\in J_H^\perp \tag{3.16}$$

which does not contain already the operator $\mathring{C}^{-1}$. It follows from the equality (3.10) for $w=w_0$ indeed

$$\mathring{C}^{-1}\mathscr{L}_1(\mathring{\psi}w_0)+[\mathrm{i}\{\mathring{\psi}_p, \nabla\}-\{\mathring{\psi}_x, \nu\}]w_0\in J_H.$$

According to the lemma 2

$$\mathscr{L}_1\mathring{C}^{-1}\mathscr{L}_1(\mathring{\psi}w_0)+\mathscr{L}_1[\mathrm{i}\{\mathring{\psi}_p, \nabla\}-\{\mathring{\psi}_x, \nu\}]w_0\in \mathscr{L}_1 J_H\subset J_H^\perp$$

which proves the equivalence of formulae (3.15) and (3.16). Substituting the operators $\mathscr{L}_1$ and $\mathscr{L}_2$ in (3.16) and using the identity (3.9) we obtain

$$\begin{aligned}Z\equiv{}&\mathring{B}\mathring{\psi}\mathscr{L}w_0+2\mathrm{i}\mathring{B}\frac{\mathrm{d}\mathring{\psi}}{\mathrm{d}\tau}w_0+\frac{\mathrm{i}}{\sqrt{g}}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathring{B}\sqrt{g})\mathring{\psi}w_0\\&+\mathrm{i}[\tfrac{1}{2}(\mathring{C}_{p_s}\mathring{\psi}_{x^s}-\mathring{C}_{x^s}\mathring{\psi}_{p_s})+\mathring{M}\mathring{\psi}]w_0 \qquad Z\in J_H^\perp.\end{aligned} \tag{3.17}$$

In particular the vector Z had to be orthogonal to the vector $\mathring{\psi}$ because $\mathring{\psi}\in J_H^\perp$. The vector in the square brackets in (3.17) is obviously orthogonal to $\mathring{\psi}$ because of (3.7) and (3.9). Hence the $w_0(\tau, \nu)$ function should satisfy the equation

$$(\mathring{B}\mathring{\psi}, \mathring{\psi})\mathscr{L}w_0+\frac{\mathrm{i}}{\sqrt{g}}\frac{\mathrm{d}}{\mathrm{d}\tau}[(\mathring{B}\mathring{\psi}, \mathring{\psi})\sqrt{g}]w_0=0. \tag{3.18}$$

It is possible to normalise the vector $\mathring{\psi}(\tau)$ in an arbitrary way at the expense of correspondingly changing the function $w_0(\tau, \nu)$ (3.11). It is convenient to set

$$\mathring{\psi}=\frac{1}{4\sqrt{g}}\mathring{\xi} \qquad \mathring{\xi}\in I_H \qquad (\mathring{B}\mathring{\xi}, \mathring{\xi})=1. \tag{3.19}$$

Then equation (3.18) and condition (3.17) take the form

$$\mathscr{L}w_0=0 \tag{3.20}$$

$$2\mathring{B}\frac{\mathrm{d}\mathring{\xi}}{\mathrm{d}\tau}+\frac{\mathrm{d}\mathring{B}}{\mathrm{d}\tau}\mathring{\xi}+[\tfrac{1}{2}(\mathring{C}_{p_s}\mathring{\xi}_{x^s}-\mathring{C}_{x^s}\mathring{\xi}_{p_s})+\mathring{M}\mathring{\xi}]\in \mathring{I}_H^\perp. \tag{3.21}$$

If $H(x,p)$ is a simple root of the equation (2.4), then the subspace I_H is one-dimensional, the vector $\mathring{\xi}$ is defined by conditions (3.19) uniquely, and (3.21) turns into an identity. In the case when the dimension of the subspace I_H is larger than unity, the condition (3.21) makes it possible to obtain the vector $\mathring{\xi}(\tau)$ uniquely if initial data $\mathring{\xi}(\tau_0)$

are known. It is not difficult to show that the condition (3.21) is equivalent to the system

$$\frac{dx}{d\tau} = \sum_{l=1}^{r} N_{sl}(\tau)\alpha_l \qquad s = 1, \ldots, r \tag{3.22}$$

for the coordinates $(\alpha_1, \ldots, \alpha_r)$ of the vector $\mathring{\xi}(\tau)$ in orthonormal basis of the subspace $\mathring{I}_H$ where the scalar product is defined by the equality

$$\langle \xi, \psi \rangle = (B\xi, \psi).$$

The matrix $N_{sl}(\tau)$ is antisymmetric therefore the expression

$$\sum_{s=1}^{r} \alpha_s^2 = \langle \mathring{\xi}, \mathring{\xi} \rangle = (\mathring{B}\mathring{\xi}, \mathring{\xi})$$

is the first integral of the system (3.22) in agreement with the normalisation condition (3.19) of ξ.

4. Solutions of the parabolic equation

The equation

$$\mathscr{L}G = 0 \tag{4.1}$$

where the operator $\mathscr{L}$ is defined by the equality (3.8) is usually called parabolic equation in the diffraction theory. In fact $\mathscr{L}$ is a Schrödinger operator on the transversal stratification $\{\tau, \phi_S(\tau)\}$ of the ray S'. It is important that the matrix coefficients of the canonical system in the linear approximation (2.12) coincide. The mentioned coincidence makes it possible (as well as in the case of Laplace operator) to construct a rather wide class of the parabolic equation solutions if solutions of the canonical system (2.12) are known. We are interested in solutions of the equation (4.1) decreasing for $|\nu| \to \infty$.

Let us take

$$G_0(\tau, \nu) = a(\tau) \exp[\tfrac{1}{2}(\Gamma(\tau)\nu, \nu)] \tag{4.42}$$

where $a(\tau)$ is a scalar function and $\Gamma(\tau)$ is a symmetric $n \times n$ matrix with the positive imaginary part

$$\Gamma = \Gamma^{\mathrm{T}} \qquad \operatorname{Im} \Gamma > 0. \tag{4.3}$$

Substituting (4.2) in equation (4.1) we obtain the system of equations for a and Γ:

$$(\mathrm{d}a/\mathrm{d}\tau) + \tfrac{1}{2}a \operatorname{Tr}(R\Gamma + L) = 0 \tag{4.4}$$

$$(\mathrm{d}\Gamma/\mathrm{d}\tau) + \Gamma R \Gamma + \Gamma L + L^{\mathrm{T}}\Gamma = 0. \tag{4.5}$$

Let the matrices Y and Q of order $n \times n$ depending on τ satisfy the canonical system in the linear approximation

$$(\mathrm{d}/\mathrm{d}\tau)Y = RQ + LY \qquad (\mathrm{d}/\mathrm{d}\tau)Q = -L^{\mathrm{T}}Q - \mathrm{T}Y \tag{4.6}$$

and the conditions

$$Y^{\mathrm{T}}Q - Q^{\mathrm{T}}Y = 0 \qquad Y^{+}Q - Q^{+}Y = \mathrm{i}E. \tag{4.7}$$

Here E is the unity matrix, and $^{+}$ denotes the Hermitian conjugation. Because the left parts of the equalities (4.7) are first integrals of the system (4.6) it is always possible to satisfy conditions (4.7) at the expense of a suitable choice of initial data for the matrices Y and Q.

One can directly check that the matrix $\Gamma = QY^{-1}$ is a solution of the equation (4.5). Conditions (4.3) are also fulfilled:

$$\Gamma - \Gamma^{\mathrm{T}} = QY^{-1} - Y^{\mathrm{T}-1}Q^{\mathrm{T}} = Y^{\mathrm{T}-1}(Y^{\mathrm{T}}Q - Q^{\mathrm{T}}Y)Y^{-1} = 0$$

$$\operatorname{Im}\Gamma = (1/2\mathrm{i})(\Gamma - \Gamma^{+}) = (1/2\mathrm{i})\,Y^{+-1}(Y^{+}Q - Q^{+}Y)Y^{-1} = \tfrac{1}{2}Y^{+-1}Y^{-1} > 0.$$

We multiply the first of the equations (4.6) by Y^{-1}

$$(\mathrm{d}Y/\mathrm{d}\tau)\,Y^{-1} = RQY^{-1} + L = R\Gamma + L.$$

Using the last equality and the well known identity

$$(\det Y)^{-1}\,(\mathrm{d}/\mathrm{d}\tau)\det Y = \operatorname{Tr}(\mathrm{d}Y/\mathrm{d}\tau)\,Y^{-1}$$

equation (4.4) can be easily integrated:

$$a(\tau) = \text{constant}\,(\det Y(\tau))^{-1/2}.$$

One can prove (Babich and Buldyrev 1972), that according to the equalities (4.7)

$$\det Y(\tau) \neq 0.$$

Therefore the function

$$G_0(\tau, \nu) = (\det Y)^{-1/2} \exp[\tfrac{1}{2}\mathrm{i}(QY^{-1}\nu, \nu)] \tag{4.8}$$

is a solution of the parabolic equation (4.1) decreasing at $|\nu| \to \infty$.

It is possible to construct the sequence of solutions G_m of the parabolic equation proceeding from the solution G_0 with the help of so-called creation and destruction operators

$$\begin{aligned} \Lambda_j^{+} &= \mathrm{i}^{-1}(Y_j^{*}, \nabla) - (q_j^{*}, \nu) \\ \Lambda_j &= \mathrm{i}^{-1}(y_j, \nabla) - (q_j, \nu) \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{4.9}$$

Here Y_j and q_j denote columns of the matrices Y and Q, and an asterisk the complex conjugation.

The commutation equalities follow from (4.6) and (4.7)

$$[\Lambda_j, \Lambda_k] = 0 \qquad [\Lambda_j, \Lambda_k^{+}] = \delta_{jk} \tag{4.10}$$

$$[\mathscr{L}, \Lambda_j] = 0 \qquad [\mathscr{L}, \Lambda_j^{+}] = 0 \qquad j, k = 1, 2, \ldots, n \tag{4.11}$$

where δ_{jk} is the Kronecker symbol. Applying the operators Λ_j to the function $G_0(\tau, \nu)$ we get zero because

$$\Lambda_j G_0 = \Lambda_j\{a \exp[\tfrac{1}{2}\mathrm{i}(\Gamma\nu, \nu)]\} = a \exp[\tfrac{1}{2}\mathrm{i}(\Gamma\nu, \nu)][(\Gamma y_j, \nu) - (q_j, \nu)] = 0. \tag{4.12}$$

According to (4.11) the functions

$$G_m = G_{m_1 \ldots m_n} = (\Lambda_1^{+})^{m_1} \ldots (\Lambda_n^{+})^{m_n} G_0 = P_m(\tau, \nu) \exp[\tfrac{1}{2}\mathrm{i}(\Gamma\nu, \nu)]$$

are solutions of the parabolic equation (4.1). Here $m = (m_1, \ldots, m_n)$, m_i are non-negative integers, $P_m(\tau, \nu)$ are polynomials in ν of order $|m| = m_1 + \ldots + m_n$ with coefficients depending upon τ. The linear independence of the functions G_m follows from the orthogonality relations

$$\int_{\phi_S(\tau)} G_m G_m^* \, d\nu = (2\pi)^{n/2} m! \delta_{m,m}$$

which can be easily checked with the help of integration by parts and equalities (4.10), (4.12).

It can be proved that $G_m(\tau, \nu)$ are eigenfunctions of the elliptic operator $\Sigma_{j=1}^n \Lambda_j^+ \Lambda_j$ on $\phi_S(\tau)$ with eigenvalues equal to $|m|$ at fixed $\tau^{j=1}$. The functions G_m form a complete basis in $L_2(\phi_S(\tau))$.

Thus we have constructed in the zeroth approximation a sequence of formal asymptotic solutions

$$U_{0,m}(\tau, \nu) = \overset{\circ}{\xi}(\tau) g(\tau)^{-1/4} e^{i\omega\tau} G_m(\tau, \nu)$$

of the initial system (2.1) which are concentrated in a small vicinity (of $\omega^{-1/2}$ order of magnitude) of the ray S' and decrease exponentially outside of this vicinity.

The recurrent system (3.4) allows in principle to obtain the subsequent terms of the formal asymptotic expansion (3.2) as well.

In conclusion we shall mention that the method is also good for the construction of localized solutions for scalar elliptic equations on a Riemannian manifold.

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